

Theoretical Aspects on The Mathematical Basis of the GAUSS-KRÜGER and 1970 STEREOGRAPHIC Projections

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In Romania the most used projections for the map products are Gauss-Krüger and 1970 Stereographic.

The paper deals with the two projections from mathematical cartography point of view, as follows: definitions and main features, defining of the geodetic and rectangular coordinates computation, convergence of meridians angle, linear distortion coefficient and of the distances formulas.

The Gauss-Krüger projection is described from as belonging to the conformal transverse cylindrical projections group. To get rectangular and geodetic coordinates computation formulas within the paper two functions are defined. The first allows the point by point representation of an ellipsoid on a plan (the direct problem) and, the latter, the obtaining of the curvilinear coordinates of the points on the ellipsoid (the inverse problem).

The convergence of meridians angle, linear distortion coefficient and the distances are defined by solving these two problems.

Further on, the paper refers to map identifications for this type of projection.

1970 Stereographic projection is described as any azimuthal conformal projection with some specific features for the Romanian territory.

Defining of the computation formulas of this projection is also made within the direct and inverse problem solving.

I GAUSS-KRÜGER PROJECTION

1. The definition of the projection system

The GAUSS-KRÜGER projection belongs to the group of the cylindrical transversal conformal projections, which give the reference ellipsoid representation straight on a cylinder plan. The surface of ellipsoid is represented on a plane after being divided into limited meridian zones (slices) with a 6° , or 3° width.

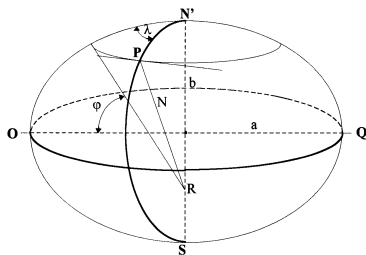


Figure 1

The projection of the ellipsoid's surface on a plane is made in the following terms:

- representation must be conform (orthometric), i.e. the angles are to be kept undistorted;
- the representation of the central meridian of a slice is a straight line in report to which the projection is symmetrical;
- the scale on the axial meridian direction is $K_0=1$, this means that the cylinder which

covers the ellipsoid is tangent at the axial meridian.

The meridians and the parallels are represented by random curves. The meridians are symmetrical opposite to the central meridian of the slice, which, according to the first condition, are represented by straight lines. The parallels are symmetrical opposite to the equator and are represented by straight lines.

The position of a point P in the plane of the projection can be determined in a system of Cartesian coordinates X and Y. The X axis coincides with the axial meridian of the slice pointed to the North, and the Y axis coincides with the equator parallel pointed to the East

It is used the Krasovsky ellipsoid associated with the Geodetic Reference System 1942 (S42).

semi-major axis $a = 6\,378\,245,0$ m; semi-minor axis $b = 6\,356\,863,0188$ m.

One can consider the following derived parameters of the ellipsoid:

- the first eccentricity: $e^2 = \frac{a^2 - b^2}{a^2}$

- the second eccentricity: $e'^2 = \frac{a^2 - b^2}{b^2}$

- the major normal for a given latitude φ : $N = \frac{a}{\sqrt{1 - e^2 \sin^2 \varphi}}$

- the curvature radius of the φ latitude parallel: $B = N \cos \varphi$

- the curvature radius of the φ latitude meridian: $M = \frac{N(1 - e^2)}{1 - e^2 \sin^2 \varphi}$

- the meridian arc (β) between φ_1 and φ_2 parallels: $\beta = \int_{\varphi_1}^{\varphi_2} M d\varphi$

The position of a random point P on the surface of the reference ellipsoid can be determined by the ellipsoidal or geodetic coordinates.

2. Defining of the transformation functions

A point P on the ellipsoid is the affix of a complex number $\omega = u + iv$, determined by its components u and v or by the curvilinear coordinates in an ellipsoidal or geodetic rectangular reference system. In the same way, a point P' on the plane is the affix of a complex number $z = x + iy$, the Cartesian coordinated in an axis system, or defined by polar coordinates.

In this way a certain mathematical biunivocal function can be defined, by which to the u and v values, which define a point P on the ellipsoid, must correspond the (x, y) , variables which determine a point P' on the plane (P' is the image of P).

The mathematical mapping has two problems to be solved:

- defining a differential function (ω) which allows the point by point representation of an ellipsoid on a plan (the direct problem);
- defining a differential function $\omega=F(z)$ which allows the obtaining of the curvilinear coordinates (u,v) of the P point on the ellipsoid, if the Cartesian coordinates (x,y) of the P point projection on the plan are known (the inverse problem).

Further on, the method of defining the two transformation functions (direct f , inverse F) in the case of Gauss projection for the practical solving of the two kinds of problem, will be presented.

A complex function $z=f(\omega)$ is a biunivoc correspondence between two complex variables:

$$z=x+iy ; \quad \omega=u+iv$$

which means defining x and y like real and continuous functions by ω , i.e.:

$$\left. \begin{array}{l} x=x(u,v) \\ y=y(u,v) \end{array} \right\} z=f(\omega) \quad (2.1)$$

and hence

$$z=x+iy=f(\omega)=f(u+iv)=x(u,v)+iy(u,v) \quad (2.2)$$

Since $x(u,v)$ and $y(u,v)$ have continuous derivative in u and v , not in the same time equal with 0, the necessary and sufficient condition for the derivative in $\omega=u+iv$ to exist, hence for the function $z=f(\omega)$ to be analytical, is the following relation being:

$$f' = \frac{\partial x}{\partial u} + i \frac{\partial y}{\partial u} = \frac{\partial y}{\partial v} - i \frac{\partial x}{\partial v} \quad \text{or the relations} \quad \begin{cases} \frac{\partial x}{\partial u} = \frac{\partial y}{\partial v} \\ \frac{\partial y}{\partial u} = -\frac{\partial x}{\partial v} \end{cases} \quad (2.3)$$

The relations are called D'Alembert or Cauchy-Riemann's equations. The functions (2.1) define the ellipsoid-plane transformation, in which $u=\lambda$, $v=\varphi$ are geodetic coordinates. The equation $z=f(\alpha)$ is an orthometric correspondence if the function $f(\alpha)$ is analytical; then, (x,y) and (u,v) are called isometric coordinates.

In the same way one can define the inverse transformation:

$$\omega=u+iv=F(z)=F(x+iy)=u(x,y)+iv(x,y) \quad (2.4)$$

i.e., can unique determinate:

$$u=u(x,y); v=v(x,y) \quad (2.5)$$

in which x,y are Cartesian coordinates.

These considerations give the possibility for defining the functions (2.2) and (2.4) with a wished approximation, by a Taylor expanding in power series around the elements of symmetrical axis (e.g., central meridian of a slice).

On the basis of the first defining condition for Gauss projection (to be orthometric) the analytical correspondences (2.2), (2.4) one can obtain :

$$\left\{ \begin{array}{l} z = f(\omega) = x + iy \\ \omega = F(z) = q + i\lambda \end{array} \right. \quad (2.6)$$

$$\left\{ \begin{array}{l} z = f(\omega) = x + iy \\ \omega = F(z) = q + i\lambda \end{array} \right. \quad (2.7)$$

which define two functions by two series with constant coefficients, determined by the second condition (the longitude preserved by transformation in report to the central meridian).

Hence, the function (2.6) solves the direct problem and the function (2.7) solves the inverse problem).

3.The Gauss projection transformation function

In figure 2: O - the origin of the Gauss Cartesian system; λ_0 - the central meridian longitude; P(φ, λ) - a random point by φ and λ coordinates; X_0 - the ordinate of φ_0 latitude parallel; β - the ordinate of φ latitude parallel;

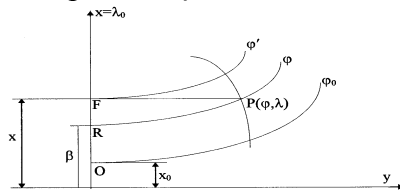


Figure 2

R - the intersection point the of the central meridian and the φ parallel; F - a point on the central meridian with y ordinate;

Let us consider :

$$\Delta\lambda = \lambda - \lambda_0; \Delta\varphi = \varphi - \varphi_0;$$

$$\Delta q = q - q_0; \Delta x = x - x_0;$$

$$\Delta z = z - z_0; \Delta\omega = \omega - \omega_0$$

then:

$$\Delta z = f(\Delta\omega) = \Delta x + iy \quad (2.8)$$

$$\Delta\omega = F(\Delta z) = \Delta q + i\Delta\lambda \quad (2.8')$$

Expanding in MacLaurin power series we obtain:

$$\Delta z = \Delta x + i\Delta y = f(\Delta q + i\Delta\lambda) = a_1(\Delta q + i\Delta\lambda) + a_2(\Delta q + i\Delta\lambda)^2 + a_3(\Delta q + i\Delta\lambda)^3 + \dots \quad (2.9)$$

$$\Delta\omega = \Delta q + i\Delta\lambda = F(\Delta x + i\Delta y) = b_1(\Delta x + i\Delta y) + b_2(\Delta x + i\Delta y)^2 + b_3(\Delta x + i\Delta y)^3 + \dots \quad (2.10)$$

in which:
$$a_n = \frac{1}{n!} \frac{d^n(x + iy)}{d(q + i\Delta\lambda)^n} \quad (2.11)$$

$$b_n = \frac{1}{n!} \frac{d^n(q + i\Delta\lambda)}{d(x + iy)^n} \quad (2.12)$$

The characteristics of the analytical functions allow to obtain the derivative independent of the direction. Hence, we choose the central meridian direction ($\Delta\lambda = \lambda - \lambda_0 = 0$), uniquely defined by ($x = \beta$), and then we can obtain the expression of the partial derivate.

Accordingly, the coefficients a_n and b_n will be:

$$a_n = \frac{1}{n!} \frac{d^n z}{d\omega^n} = \frac{1}{n!} \frac{d^n \beta}{dq^n} \quad (2.13)$$

$$b_n = \frac{1}{n!} \frac{d^n \omega}{dz^n} = \frac{1}{n!} \frac{d^n q}{d\beta^n} \quad (2.14)$$

Taking into account that

$$d\beta = M d\varphi$$

$$\frac{N}{M} = 1 + e^{\eta^2} \cos^2 \varphi = 1 + \eta^2 \quad dq = \frac{M}{N \cos \varphi} d\varphi$$

the a_n coefficients for $\frac{d^n \beta}{dq^n}$ will be:

$$\begin{aligned}
a_1 &= N \cos \varphi \quad ; a_2 = -\frac{1}{2} N \cos^2 \varphi \operatorname{tg} \varphi \quad ; a_3 = -\frac{1}{6} N \cos^3 \varphi (1 - \operatorname{tg}^2 \varphi + \eta^2) \quad ; \\
a_4 &= \frac{1}{24} N \cos^4 \varphi \operatorname{tg} \varphi (5 - \operatorname{tg}^2 \varphi + 9\eta^2 + 4\eta^4) \quad ; \\
a_5 &= \frac{1}{120} N \cos^5 \varphi (5 - 18\operatorname{tg}^2 \varphi + \operatorname{tg}^4 \varphi + 14\eta^2 - 58\eta^2 \operatorname{tg}^2 \varphi + \dots) \quad ; \\
a_6 &= -\frac{1}{720} N \cos^6 \varphi \operatorname{tg} \varphi (61 - 58\operatorname{tg}^2 \varphi + \operatorname{tg}^4 \varphi + 270\eta^2 - 320\eta^2 \operatorname{tg}^2 \varphi + \dots) \quad ;
\end{aligned}$$

and b_n coefficients for $\frac{d^n \beta}{d\varphi^n}$ will be:

$$\begin{aligned}
b_1 &= \frac{1}{N \cos \varphi} \quad ; \quad b_2 = \frac{1}{2N^2 \cos \varphi} \operatorname{tg} \varphi \quad ; \quad b_3 = \frac{1}{6N^3 \cos \varphi} (1 + 2\operatorname{tg}^2 \varphi + \eta^2) \quad ; \\
b_4 &= \frac{1}{24N^4 \cos \varphi} \operatorname{tg} \varphi (5 + 5\operatorname{tg}^2 \varphi + \eta^2 - 4\eta^4) \quad ; \\
b_5 &= \frac{1}{120N^5 \cos \varphi} (5 + 20\operatorname{tg}^2 \varphi + 24\operatorname{tg}^4 \varphi + 6\eta^2 + 8\eta^4 \operatorname{tg}^2 \varphi + \dots) \quad ; \\
b_6 &= \frac{1}{720N^6 \cos \varphi} \operatorname{tg} \varphi (61 + 180\operatorname{tg}^2 \varphi + 120\operatorname{tg}^4 \varphi + 46\eta^2 + 48\eta^4 \operatorname{tg}^2 \varphi + \dots) \quad ;
\end{aligned}$$

4. The evaluation of the Gauss Cartesian coordinates from the geodetic coordinates (direct problem)

Shifting origin O in R (in figure 2) and having given φ and λ of point P, one may evaluate their Cartesian coordinates (x,y) in Gauss projection by (2.9) and (2.13):

$$\begin{cases} x = a_0 + a_2 p^2 + a_4 p^4 + A_6 \\ y = b_1 p + b_3 p^3 + B_5 \end{cases} \quad (3.1)$$

Taking into account that:

$$p = (\lambda - \lambda_0)'' \cdot 10^{-4}$$

e, e', a -the reference ellipsoid parameters

A, B, C, D, E -the coefficients which determine the meridian arc on the ellipsoid

$$A = a \cdot (1 - e^2) \cdot \left(1 + \frac{3}{4} e^2 + \frac{45}{64} e^4 + \frac{350}{512} e^6 + \frac{11025}{16384} e^8\right) \quad ; \quad B = \frac{a \cdot (1 - e^2)}{2} \cdot \left(\frac{3}{4} e^2 + \frac{60}{64} e^4 + \frac{525}{512} e^6 + \frac{2205}{2048} e^8\right)$$

$$C = \frac{a \cdot (1 - e^2)}{4} \cdot \left(\frac{15}{64} e^4 + \frac{210}{512} e^6 + \frac{2205}{4096} e^8\right) \quad ; \quad D = \frac{a \cdot (1 - e^2)}{6} \cdot \left(\frac{35}{512} e^6 + \frac{315}{2048} e^8\right) \quad ; \quad E = \frac{a \cdot (1 - e^2)}{8} \cdot \frac{315}{16384} e^8$$

$$\beta = A \cdot \varphi - B \sin 2\varphi + C \sin 4\varphi - D \sin 6\varphi + E \sin 8\varphi$$

$$\sin 1'' = \frac{1}{\rho''} \quad (3.2)$$

K_0 – scale factor; $K_0=1$

$$N = \frac{a}{\sqrt{1 - e^2 \sin^2 \varphi}} \quad ; \quad \eta = e' \cdot \cos \varphi$$

One can define the coefficients $a_0, a_2, a_4, A_6, b_1, b_3, B_5$ thus:

$$a_0 = K_0 \cdot \beta \quad ; \quad a_2 = \frac{1}{2} N \cdot \cos^2 \varphi \cdot \operatorname{tg} \varphi \cdot \sin^2 1'' \cdot K_0 \cdot 10^8 \quad ; \quad a_4 = \frac{1}{24} N \cdot \cos^4 \varphi \cdot \operatorname{tg} \varphi \cdot \sin^4 1'' \cdot (5 - \operatorname{tg}^2 \varphi + 9\eta^2 + 4\eta^4) \cdot K_0 \cdot 10^{16}$$

$$A_6 = \frac{1}{720} p^6 N \cos^6 \varphi \cdot \operatorname{tg} \varphi \cdot \sin^6 1'' (61 - 58\operatorname{tg}^2 \varphi + 270\eta^2 - 320\eta^2 \operatorname{tg}^2 \varphi) \cdot K_0 \cdot 10^{24}$$

$$b_1 = N \cdot \cos \varphi \cdot \sin 1'' \cdot K_0 \cdot 10^4 \quad ; \quad b_3 = \frac{1}{6} N \cos^3 \varphi \cdot \sin^3 1'' \cdot (1 - \operatorname{tg}^2 \varphi + \eta^2) \cdot K_0 \cdot 10^{12}$$

$$B_5 = \frac{1}{120} p^5 \cdot N \cos^5 \varphi \cdot \sin^5 1'' \cdot (5 - 18\operatorname{tg}^2 \varphi + 18\operatorname{tg}^4 \varphi + 14\eta^2 - 58\eta^2 \operatorname{tg}^2 \varphi) \cdot K_0 \cdot 10^{20}$$

By agreement adopted, the meridian abscissa is set to $y_0 = 500\,000$ m, for avoiding negative values for points placed in the west side of central meridian of the slice. The same, by agreement adopted, the Equator ordinate is set to $x_0 = 10\,000\,000$ m, for avoiding negative values on southern hemisphere.

Thus the final coordinates are:

$$y_1 = 500\,000 + y; \quad x_1 = x \text{ (northern hemisphere)}; \quad x_1 = 10\,000\,000 - x \text{ (southern hemisphere)} \quad (3.3)$$

For homogenizing the coordinates (x_1, y_1) ,

$$y_2 = y_1 + N_f \cdot 1\,000\,000$$

(N_f is 4 or 5 for 6° slices on Romanian territory).

5. The evaluation of the geodetic coordinate from the Gauss Cartesian coordinates (inverse problem)

Shifting the origin O in F (in figure 2) and having given (x, y) Gauss coordinates of a point P, one may evaluate the geodetic coordinates (φ, λ) with the following relations:

$$\begin{cases} \varphi = \varphi' - a_1 q^2 + a_3 q^4 - D_6 \\ \lambda = \lambda_0 + b_2 q - b_4 q^3 + E_5 \end{cases} \quad (4.1)$$

$$N' = \frac{a}{\sqrt{1 - e^2 \sin^2 \varphi'}}$$

Taking into account that: $q = (y - 500\,000) \cdot 10^{-6}$;

A, B, C, D, E - coefficients of meridian arc on ellipsoid;

$$\eta = e' \cos \varphi'$$

$K_0 = 1$ - scale factor;

λ_0, φ_0 - geodetic coordinates of central meridian;

e, e^2, a - the reference ellipsoid parameters

$x_0 = 500\,000$;

Latitude φ' is iterative evaluated

$$\varphi = (x - x_0) / A; \quad \varphi_0 = \varphi; \quad \Delta \varphi = (B \sin 2\varphi - C \sin 4\varphi + D \sin 6\varphi - E \sin 8\varphi) / A; \quad \varphi' = \varphi_0 + \Delta \varphi \quad (4.2)$$

$$|\varphi - \varphi'| > 10^{-5} \quad (4.3)$$

If then $\varphi = \varphi'$ and restarts from (4.2).

If $|\varphi - \varphi'| \leq 10^{-5}$ then the value φ' obtained by (4.3) will be used to evaluate the coefficients a_1, a_3, D_6, b_2, E_5 and the values N' and η' .

The coefficients are:

$$a_1 = \frac{tg \varphi'}{2N'^2 \sin 1''} \cdot (1 + \eta'^2) \cdot \frac{1}{K_0^2} \cdot 10^{12}$$

$$a_3 = \frac{1}{24N'^4 \sin 1''} \cdot tg \varphi' (5 + 3tg^2 \varphi' + 6\eta'^2 - 6tg^2 \varphi' \cdot \eta'^2 - 3\eta'^4 - 9tg^2 \varphi' \cdot \eta'^4) \cdot \frac{1}{K_0^6} \cdot 10^{24}$$

$$D_6 = \frac{tg \varphi'}{720N'^6 \sin 1''} \cdot q^6 (61 + 90tg^2 \varphi' + 45tg^4 \varphi' + 107\eta'^2 - 162tg^2 \varphi' \cdot \eta'^2 - 45tg^4 \varphi' \cdot \eta'^2) \cdot \frac{1}{K_0^6} \cdot 10^{36}$$

$$b_2 = \frac{1}{N' \cos \varphi' \sin 1''} \cdot \frac{1}{K_0} \cdot 10^6; \quad b_4 = \frac{1}{N' \cos \varphi' \sin 1''} \cdot (1 + 2tg^2 \varphi' + \eta'^2) \cdot \frac{1}{K_0^3} \cdot 10^{18}$$

$$E_5 = \frac{1}{6N'^3 \cos \varphi' \sin 1''} \cdot q^5 (5 + 28tg^2 \varphi' + 24tg^4 \varphi' + 6\eta'^2 + 8tg^2 \varphi' \cdot \eta'^2) \cdot \frac{1}{K_0^3} \cdot 10^{30}$$

6. The meridians convergence angle (γ)

The meridians convergence angle is the angle γ between the meridian projection and a parallel to ordinate in the point P(x,y). This angle can be expressed in report with geodetic coordinates or Gauss Cartesian coordinates.

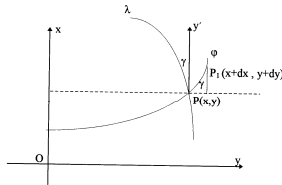


Figura 5

Figure 3

From figure 3 we deduce that $tg\gamma = \frac{dx}{dy} = \frac{dx}{d\lambda} \cdot \frac{d\lambda}{dy}$. Doing the differentiation of (3.1) in report with λ , we obtain the meridian convergence γ :

$$\gamma' = \gamma_1 p + \gamma_2 p^3 + C_5$$

in which: γ' - the meridian convergence expressed in arc seconds;

$$p = 0.0001 \Delta\lambda'' = 10^{-4} \cdot (\lambda - \lambda_0); \quad \gamma_1 = \sin\varphi \cdot 10^4; \quad \gamma_2 = \frac{1}{2} \cdot \sin\varphi \cdot \cos^2\varphi \cdot (1 + 3\eta^2 + 2\eta^4) \cdot \sin^2 1'' \cdot 10^{12}$$

$$C_5 = \frac{1}{15} \cdot \sin\varphi \cdot \cos^4\varphi \cdot (2 - tg^2\varphi) \cdot p^5 \cdot \sin^4 1'' \cdot 10^{20}; \quad \eta = e' \cdot \cos\varphi$$

λ_0 = central meridian longitude of the slice.

The meridian convergence in report with Gauss Cartesian coordinates (\bar{x}, \bar{y}), is expressed thus:

$$\gamma' = g_1 \cdot q - g_2 \cdot q + F_5 \quad (5.2)$$

in which: $q = (x - 500\,000)10^{-6}$

$$g_2 = \frac{12W_{12} \cdot 2^{11} I_w}{18\phi_1} \cdot (5 + 218_5\phi_1 + 318_4\phi_1) \cdot d_2 \cdot \frac{K_2^0}{1} \cdot 10_{30}$$

$$g_1 = \frac{W_{12} \cdot 2^{11} I_w}{18\phi_1} \cdot \frac{K_0^0}{1} \cdot 10_6; \quad g_5 = \frac{3W_{13} \cdot 2^{11} I_w}{18\phi_1} \cdot (1 + 18_5\phi_1 - 11_5 - 511_4) \cdot \frac{K_3^0}{1} \cdot 10_{18}$$

ϕ' - the meridian arc on the ellipsoid, iterative evaluated as shown in II.4;

$$N = \frac{a}{\sqrt{1 - e'^2 \sin^2 \phi'}}; \quad (5.3) \quad \eta = e' \cos\phi'; \quad (5.4) \quad K_0 = 1 \text{ - scale factor.}$$

7. The coefficient of linear distortion (K)

The coefficient of linear distortion of a point P(φ, λ), in direct problem is evaluated by:

$$K = K_0 \left(1 + b_5 \cdot \frac{5}{1} \cos^2\phi \cdot 2^{11} I_w (1 + 11_5) \cdot 10_8 \right) \quad (6.1)$$

in which $p = 10^{-4} (\lambda - \lambda_0)''$; $\eta = e' \cos\varphi$; λ_0 - the longitude of the central meridian;

$K_0 = 1$ - scale factor

$$K = K_0 (1 + k_1 \cdot q^2 + 3 \cdot 10^5 q^4)$$

The coefficient of linear distortion of a point P(x,y), in inverse problem is evaluated by

$$q = 10^{-6} (y - 500000); \quad k_1 = \frac{1}{2N'^2} \cdot (1 + \eta^2) \cdot \frac{1}{K_0^2} \cdot 10^{12} \cong \frac{10^{12}}{2R^2} \quad (6.2)$$

in which N, η' - see II.6 ((5.3),(5.4)); $K_0=1$ scale factor; R - mean curvature ray for the latitude φ .

8. The evaluation of the relative distance to the ground

The linear Gauss values are distorted with a certain value. This value is given by the projection distortion. To determine a certain distance to the ground with the Gauss Cartesian coordinates we must take into account the linear distortion.

Using:

$$S = \frac{D}{K} = \frac{\sqrt{\Delta x^2 + \Delta y^2}}{K}; \quad \text{The distance value on the ground.}$$

$$\frac{1}{K} = \frac{1}{6} \cdot \left(\frac{1}{K_1} + \frac{4}{K_m} + \frac{1}{K_2} \right) \quad \text{The mean inverse coefficient of linear distortion}$$

K_1, K_2, K_m

$$K_i = K_0 K'_i = K_0 \left[1 + \frac{1}{2N_i'^2} (1 + \eta_i'^2) \cdot \frac{1}{K_0^2} \cdot 10^{12} \cdot q^2 + 0.00003 \cdot q^4 \right]; \quad K_0 = 1 \quad q = x \cdot 10^{-6} \quad N = \frac{1}{\sqrt{1 - e^2 \sin^2 \varphi'_i}}$$

The distortion coefficients in end points and mean point.

One can compute for K_m

$$q_m = \frac{(y_1 + y_2)}{2} \cdot 10^{-6}; \quad \varphi'_m = \frac{\varphi'_1 + \varphi'_2}{2}; \quad N'_i = \frac{a}{\sqrt{1 - e^2 \sin^2 \varphi'_i}}; \quad \eta'_i = e' \cos \varphi'_i; \quad e'^2 = \frac{a^2 - b^2}{b^2}$$

a, b - semi-axis of the reference ellipsoid

φ' - the latitude of the meridian arc $X=j$ evaluated in Chap. 4 ((4.2)(4.3)).

II. STEREOGRAPHIC PROJECTION 1970

1. The definition of the projection system

The stereographic projection is an azimuthal orthometric projection. The reference ellipsoid is represented on a tangent plane in central zone point of the interested territory or to a secant plane using a secant circle (Romanian territory).

The geometrical representation is

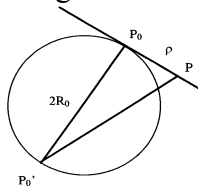


Figure 4

- The Earth is a sphere with R_0 radius (mean curvature radius);
- P_0 is central zone point for mapping projection (projection pole);
- From P'_0 (P_0 antipode) on represent all the sphere points on a tangent projection plane in P_0 or on a secant plane.

The principal elements of stereographic projection plane are:

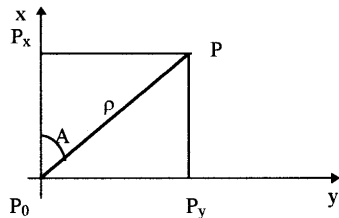


Figure 5

The distance $PP_0 = \rho$ (defined by Roussille) is given by: $\rho = 2R_0 \arctan \frac{X - X_0}{2R_0}$. In which X is the parallel of the point P and X_0 is the parallel of the point P_0 . Thus the Cartesian stereographic coordinates of the point P are:

$$x = \rho \cos A; y = \rho \sin A$$

Now in Romania the Stereographic 1970 projection is used, which has Făgăraș town as the pole of projection. Its geographic coordinates are:

$$\varphi_0 = 46^\circ \text{ (North); } \lambda_0 = 25^\circ \text{ (East)}$$

The scale factor is $V = 0,99975$.

The projection is applied on a secant plan with a minor circle (zenithal distance $1^\circ,4843$). It is used the Krasovsky ellipsoid (see II.1).

2. The evaluation of the cartesian stereographical coordinates from the geodetic coordinates

Let us consider a point $P(\varphi, \lambda)$ on the ellipsoid, $p(x, y)$ its projection on a plan and q the isometric latitude of the point P . Denote $\varphi_0, \lambda_0, q_0$ the pole projection coordinates. On the basis of Cauchy-Riemann (chap II.2) conditions, x and y must be an analytical function in report with $q + il$: $x + iy = f(q + il)$. If denote: $\Delta\varphi = \varphi - \varphi_0, l = \lambda - \lambda_0, \Delta q = q - q_0$ and (x_0, y_0) are the pole projection stereographic coordinates, then: $x + iy = f[(q_0 + i\lambda_0) + (\Delta q + il)]$ (7.1)

If the pole projection is the origin of the rectangular coordinates, then Taylor expanding in power series (7.1) around the pole projection, one obtain:

$$x + iy = a_1(\Delta q + il) + a_2(\Delta q + il)^2 + a_3(\Delta q + il)^3 + \dots$$

Computing and separating the real side from the imaginary side, then:

$$\begin{cases} \lambda = \alpha^1 + 5\alpha^3 \nabla^2 \nabla^2 l + 3\alpha^3 \nabla^3 \nabla^2 l - \alpha^3 l^3 + \alpha^4 \nabla^4 \nabla^2 l - \alpha^4 \nabla^4 l^3 + 2\alpha^2 \nabla^4 \nabla^2 l - 10\alpha^2 \nabla^4 \nabla^2 l^3 + \dots \\ x = \alpha^1 \nabla^1 \nabla^1 + \alpha^2 \nabla^2 \nabla^2 - \alpha^2 l^2 + \alpha^3 \nabla^3 \nabla^2 - 3\alpha^3 \nabla^3 \nabla^2 l^3 + \alpha^4 \nabla^4 \nabla^2 - \alpha^4 \nabla^4 \nabla^2 l^3 + \alpha^4 \nabla^4 l^3 + \alpha^2 \nabla^2 \nabla^2 - \dots \end{cases} \quad (7.2)$$

in which: $a_n = \frac{1}{n!} \left(\frac{d^n x}{dq^n} \right)$

One replace Δq by $\Delta\varphi$, through: $\Delta q = c_1 \Delta\varphi + c_2 \Delta\varphi^2 + c_3 \Delta\varphi^3 + c_4 \Delta\varphi^4 + \dots$ (7.3)

in which: $c_n = \frac{1}{n!} \left(\frac{d^n q}{d\varphi^n} \right)$

Introducing (7.3) in (7.2) one obtain the evaluation relations for the stereographic coordinates of a point $P(\varphi, \lambda)$:

$$\begin{cases} x = x_0 + a_{10} \Delta\varphi + a_{20} \Delta\varphi^2 + a_{02} l^2 + a_{30} \Delta\varphi l^2 + a_{12} \Delta\varphi l^2 + a_{40} \Delta\varphi^4 + \dots \\ y = y_0 + b_{01} l + b_{11} \Delta\varphi l + b_{21} \Delta\varphi^2 l + b_{03} l^3 + b_{31} \Delta\varphi^3 l + b_{13} \Delta\varphi l^3 + b_{41} \Delta\varphi^4 l + b_{23} \Delta\varphi^2 l^3 + b_{51} \Delta\varphi^5 l + \dots \end{cases} \quad (7.4)$$

Denote: $\eta_0 = e' \cos \varphi_0$; $N_0 = \frac{a}{\sqrt{1 - e^2 \sin^2 \varphi_0}}$; $t_0 = tg \varphi_0$

The first coefficients (aij) and (bij) from (7.4) have the following forms:

$$\begin{aligned}
 a_{10} &= N_0(1 - \eta_0^2 + \eta_0^4 - \eta_0^6) & b_{01} &= N_0 \cos \varphi \\
 a_{20} &= \frac{3}{2} N_0 t_0 \eta_0^2 (1 - \eta_0^2) & b_{11} &= N_0 t_0 \cos \varphi_0 (1 - \eta_0^2 + \eta_0^4) \\
 a_{02} &= \frac{1}{2} N_0 t_0 \cos^2 \varphi_0 & b_{21} &= -\frac{1}{4} N_0 \cos \varphi_0 (1 - \eta_0^2 + 6t_0^2 \eta_0^2 + \eta_0^4 - 12t_0^2 \eta_0^4) \\
 a_{30} &= \frac{1}{12} N_0 (1 - 4\eta_0^2 - 6\eta_0^4 + 42t_0^2 \eta_0^2) & b_{03} &= \frac{1}{12} N_0 \cos^3 \varphi_0 (1 + 5\eta_0^2 - 6t_0^2 \eta_0^2) \\
 a_{12} &= \frac{1}{4} N_0 \cos^2 \varphi_0 (1 - 2t_0^2 + 2t_0^2 \eta_0^2 - 2t_0^2 \eta_0^4) & b_{13} &= -\frac{1}{6} N_0 t_0 \cos^3 \varphi_0 (2 - t_0^2 - 4\eta_0^2 + t_0^2 \eta_0^2) \\
 a_{40} &= \frac{1}{24} N_0 t_0 \cos^4 \varphi_0 (2 - t_0^2 + 6\eta_0^2) & & \dots\dots\dots
 \end{aligned}$$

The coefficients (7.5) are constants computed by the means of the pole projection latitude φ_0 . The stereographic Cartesian coordinate (x,y) are evaluated using (7.4) with (7.5).

For Romanian territory are enough till six order terms in (7.4).

By agreement adopted, the pole projection coordinate is set to $x_0=500\ 000$ m; $y_0=500\ 000$ m, for avoiding negative values (in the relation (7.4)).

The coordinates for P are: $x_1=500\ 000+x$; $y_1=500\ 000+y$

3.The evaluation of geodetic coordinates from stereographical coordinates

The inverse problem, to determinate the geodetic coordinates (φ, λ) from the stereographical coordinates is solved similarly with the direct problem (III.2).

Thus one can be write the following expanded in power series:

$$q+il=A_1(x+iy)+A_2(x+iy)^2+A_3(x+iy)^3+A_4(x+iy)^4+\dots \quad (8.1)$$

which became after the computation and separation of the real side from the imaginary side:

$$\begin{cases}
 q=A_1x+A_2x^2-A_2y^2+A_3x^3-3A_3xy^2+A_4x^4-6A_4x^2y^2+A_4y^4+A_5x^5-10A_5x^3y^2+\dots \\
 l=A_1y+2A_2xy+A_3x^2y-A_3y^3+4A_4xy^3+5A_5x^4y-10A_5x^2y^3+\dots
 \end{cases} \quad (8.2)$$

The coefficients A_i are: $A_n = \frac{1}{n!} \left(\frac{d^n q}{dx^n} \right)$

$$\text{Replacing } \Delta q \text{ by } \Delta \varphi, \text{ through: } \Delta \varphi=c_1 \Delta q+c_2 \Delta q^2+c_3 \Delta q^3+c_4 \Delta q^4+\dots \quad (8.3)$$

in which: $c_n = \frac{1}{n!} \left(\frac{d^n \varphi}{dq^n} \right)$

and introducing the first relation (8.2) in (8.3) and $\lambda=\lambda_0$, $\varphi=\Delta \varphi+\varphi_0$ then (8.2) became:

$$\begin{cases}
 \varphi = \varphi_0 + A_{10}x + A_{20}x^2 + A_{02}y^2 + A_{30}x^3 + A_{12}xy^2 + A_{40}x^4 + A_{22}x^2y^2 + A_{04}y^4 + A_{50}x^5 + A_{32}x^3y^2 + \dots \\
 \lambda = \lambda_0 + B_{01}y + B_{11}xy + A_{21}x^2y + B_{03}y^3 + B_{31}x^3y + B_{13}xy^3 + B_{41}x^4y + B_{23}x^2y^3 + B_{05}y^5 + B_{51}x^5y + \dots
 \end{cases} \quad (8.4)$$

Denote: $\eta = e' \cos \varphi_0$; $N_0 = \frac{a}{\sqrt{1 - e^2 \sin^2 \varphi_0}}$; $t_0 = tg \varphi_0$

The first coefficients A_{ij} and B_{ij} have the following forms:

$$\begin{aligned}
 A_{10} &= \frac{1}{N_0} (1 + \eta_0^2) & B_{01} &= \frac{1}{N_0 \cos \varphi_0} \\
 A_{20} &= -\frac{3}{2} \frac{t_0 \eta_0^2}{N_0^2} (1 + \eta_0^2) & B_{11} &= \frac{t_0}{N_0^2 \cos \varphi_0} \\
 A_{02} &= -\frac{1}{2} \frac{t_0}{N_0^2} \cos^2 \varphi_0 & B_{21} &= \frac{1}{4N_0^3 \cos \varphi_0} (1 + 4t_0^2 + \eta_0^2) \\
 A_{30} &= -\frac{1}{12} \frac{1}{N_0^3} (1 + 8\eta_0^2 - 6t_0^2 \eta_0^2 + 13\eta_0^4 - 36t_0^2 \eta_0^4) & B_{03} &= \frac{1}{12N_0^2 \cos \varphi_0} (1 + 4t_0^2 + \eta_0^2) \\
 A_{12} &= -\frac{1}{4} \frac{1}{N_0^3} (1 + 2t_0^2 + 2\eta_0^2 - 4t_0^2 \eta_0^2 - 6t_0^2 \eta_0^4) & B_{31} &= \frac{t_0}{6N_0^4 \cos \varphi_0} (3 + 6t_0^2 - \eta_0^2) \\
 A_{40} &= \frac{3}{4} \frac{t_0}{N_0^4} \eta_0^2 & B_{41} &= \frac{1}{16N_0^5 \cos \varphi_0} (1 + 12t_0^2 - 16t_0^4) \\
 A_{22} &= -\frac{1}{4} \frac{t_0}{N_0} (1 + 2t_0^2 - 8\eta_0^2 - t_0^2 \eta_0^2) & & \\
 A_{04} &= \frac{1}{24} \frac{t_0}{N_0^4} (3 + 3t_0^2 + 2\eta_0^2 - 6t_0^2 \eta_0^2) & &
 \end{aligned} \tag{8.5}$$

The coefficients (8.5) are computed constants in report with the pole projection latitude, φ_0 . The geodetic coordinates (φ, λ) , are evaluated using (7.4), with (7.5).

4. The meridians convergence angle γ

The meridians convergence angle γ , in report with geodetic coordinates (φ, λ) , in the stereographic projection can be evaluate by: $\lambda = \lambda^0 \lambda + \lambda^1 \lambda^1 \lambda + \lambda^2 \lambda^2 \lambda + \lambda^3 \lambda^3 \lambda + \dots$; in which: $l = \lambda - \lambda_0$; $\Delta \varphi = \varphi - \varphi_0$

$$\eta = e' \cos \varphi_0; \quad N_0 = \frac{a}{\sqrt{1 - e^2 \sin^2 \varphi_0}}; \quad t_0 = tg \varphi_0$$

$$\gamma_{01} = \sin \varphi_0; \quad \gamma_{03} = \frac{1}{2} t_0 \cos^3 \varphi_0 (1 + 9\eta_0^2); \quad \gamma_{11} = \frac{1}{2} \cos \varphi_0;$$

$$\gamma_{31} = \frac{1}{24} \cos \varphi_0; \quad \gamma_{21} = -\frac{5}{4} \eta_0^2 \sin \varphi_0; \quad \gamma_{13} = -\frac{1}{24} \cos^3 \varphi_0 (1 - 2t_0^2)$$

The meridian convergence in report with the stereographic coordinates can be evaluate by:

$$\lambda = \lambda^0 \lambda + \lambda^1 \lambda^1 \lambda + \lambda^2 \lambda^2 \lambda + \lambda^3 \lambda^3 \lambda + \lambda^4 \lambda^4 \lambda + \dots$$

In which:

$$\gamma'_{01} = \frac{t_0}{N_0}; \quad \gamma'_{03} = \frac{1}{12N_0^2} (3 + 4t_0^2 - 5\eta_0^2); \quad \gamma'_{11} = \frac{1}{2N_0^2} (1 + 2t_0^2 \eta_0^2); \quad \gamma'_{31} = \frac{1}{8N_0^4} (1 + 8t_0^2 + 8t_0^4);$$

$$\gamma'_{21} = \frac{t_0}{4N_0^3} (3 + 4t_0^2 - 5\eta_0^2); \quad \gamma'_{13} = -\frac{1}{8N_0^4} (1 + 8t_0^2 + 8t_0^4)$$

5. Distance reduction in the stereographic projection

The representation scale for direct problem is given by:

$$m = 1 + \frac{1}{4} \left((1 + r_0^2 + r_0^4) \Delta \varphi^2 + (1 + r_0^2)^2 + \frac{3}{4} t_0 r_0^2 \Delta \varphi^3 - \frac{1}{4} \cos^2 \varphi_0 t_0 (1 + 8r_0^2) \Delta \varphi^2 + \frac{1}{24} \Delta \varphi^4 + \frac{1}{48} \cos^4 \varphi_0 (2 - t_0^2) \varphi^4 \right)$$

and for the inverse problem:

$$m = 1 + \frac{x^2 + y^2}{4 R_0^2} - \frac{2 t_0}{R_0^3} xy \quad R_0 - \text{the mean curvature radius of the Earth.}$$

The distance d between two points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ in report with the distances on the ellipsoid can be evaluated by

$$\frac{d}{s} = 1 + \frac{x_m^2 + y_m^2}{4 R_0^2} + \frac{\Delta x^2 + \Delta y^2}{48 R_0^2},$$

in which:

$$x_m = \frac{x_1 + x_2}{2}; \quad y_m = \frac{y_1 + y_2}{2}; \quad \Delta x = x_2 - x_1; \quad \Delta y = y_2 - y_1.$$

R_0 - the mean curvature ray of the Earth

III. MAP IDENTIFICATIONS AND SHEETS NUMBERING IN THE GAUSS-KRÜGER AND 1970 STEREOGRAPHIC PROJECTIONS

The numbering sheets system is named the identification (nomenclature) of the map. This is usually printed on the north edge of each map sheet, by letters and numbers. Romania has adopted, as starting point for all scale topographic maps identification the projection used for the world international chart at 1:1,000,000 scale namely the projection of each hemisphere on a normal cone which can be easily unfolded, since 1952. According to this unfolding, the earth surface up to the proximity of the poles, has been divided onto 4° width slices by means of parallels and onto 6° width columns by means of meridians. The slices are parallel with the Equator, and the columns are placed among the meridians which are straight lines pointed to the poles.

That's why, nowadays, the Romanian topographic maps, both those in Gauss-Krüger projection and 1970 Stereographic ones, have their identifications defined within this system. The identification of the 1:1,000,000 scale map sheet consist of the Latin alphabet capital letters (from A to V) corresponding to 4° slices beginning from the Equator to the poles and of the numbers from 1 to 60, corresponding to the 6° longitude columns, beginning from the 180° meridian.

The Romanian territory is mostly covered by the L-34 and L-35 and partly by the M-34, M-35, K-34 and K-35 map sheets at 1:1,000,000 scale. The map sheet at 1:1,000,000 scale consists of 4 map sheets at 1:500,000 scale, which are printed with the Latin alphabet capital letters (A,B,C,D), 36 map sheets at 1:200,000 scale, printed with the

roman numbers (I to XXXVI) and of 144 map sheets at 1: 100,000 scale, printed with the Arabian numbers (1 to 144).

The identifications of the 1:500,000;1:200,000 and 1:100,000 scale map sheets consist of the identification of the 1:1,000 000 scale map sheet at which their proper notations are added. For the other scales, one can detail as follows: the map sheet at 1:100,000 scale consists of 4 map sheets at 1:50,000 scale, printed with the first four capital letters of the Latin alphabet (A,B,C,D); the map sheet at 1:50,000 scale consists of 4 map sheets at 1:25,000 scale, printed by the first four small letters of the Latin alphabet (a,b,c,d); the map sheet at 1:25,000 scale consists of 4 sheets of the topographic plan at 1:10,000 scale, printed with the Arabian numbers 1,2,3,4; the 1:10,000 scale topographic plan consists of 4 sheets of the topographic plan at 1:5,000 scale, printed with the first four Roman numbers (I,II,III,IV).

IV.REFERENCES

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